

# A different see-saw formula for neutrino masses

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## Abstract

In a wide class of unified models there is an additional (and possibly dominant) term in the neutrino mass formula that under the simplest assumption takes the form  $M_\nu = (M_N + M_N^T)u/M_G$ , where  $M_N$  is the neutrino Dirac mass matrix, and  $u = O(M_W)$ . This makes possible highly predictive models. A generalization of this form yields realistic neutrino masses and mixings more readily than the usual see-saw formula in some models.

In grand unified theories based on  $SO(10)$  or larger groups, right-handed neutrinos exist and typically acquire mass of order the unification scale,  $M_G \simeq 2 \times 10^{16}$  GeV. When these superheavy neutrinos are integrated out, masses are induced for the left-handed neutrinos that are of order  $v^2/M_G \simeq 10^{-3}$  eV, where  $v = \sqrt{v_u^2 + v_d^2} = (\sqrt{2}G_F)^{-1/2}$  is the weak scale. This is the so-called see-saw mechanism [1], and it well explains — at least in an order of magnitude sense — the scale of neutrino masses seen in atmospheric and solar neutrino oscillations ( $\sqrt{\delta m_{atm}^2} \simeq 5 \times 10^{-2}$  eV [2] and  $\sqrt{\delta m_{sol}^2} \simeq 8.5 \times 10^{-3}$  eV [3]).

It has been long known [4] that in certain unified schemes  $SU(2)_L$ -triplet Higgs fields exist that couple directly to the left-handed neutrinos and give them a Majorana mass of order  $v^2/M_G$ . This effect is called the type II see-saw mechanism. It typically happens, for example, in  $SO(10)$  models in which a  $\overline{\mathbf{126}}$  and  $\mathbf{126}$  of Higgs fields are responsible for breaking  $B - L$  and generating the right-handed neutrino mass matrix,  $M_R$ . In such models,

both the type I (i.e. original) and the type II see-saw mechanisms would normally operate.

In this letter we point out that in  $SO(10)$  models where a  $\overline{\mathbf{16}}$  and  $\mathbf{16}$  of Higgs fields (rather than  $\overline{\mathbf{126}}$  and  $\mathbf{126}$ ) are responsible for breaking  $B - L$  and generating  $M_R$  another type of see-saw mechanism operates that we shall call type III. The type I see-saw formula is

$$M_\nu^I = -M_N M_R^{-1} M_N^T, \quad (1)$$

where  $(M_N)_{ij} \nu_i N_j^c = (Y_N)_{ij} \nu_i N_j^c \langle H_u \rangle$  is the Dirac neutrino mass term and  $(M_R)_{ij} N_i^c N_j^c$  is the Majorana mass term of the right-handed neutrinos. Eq. (1) can be understood diagrammatically as arising from the graph in Fig. 1.

The type II see-saw formula is simply  $M_\nu^{II} = M_T$ , where  $(M_T)_{ij} \nu_i \nu_j = (Y_T)_{ij} \nu_i \nu_j \langle T \rangle$  is the direct Majorana mass term of the left-handed neutrinos coming from their coupling to the triplet Higgs field  $T$ . In  $SO(10)$ , if  $M_R$  comes from the Yukawa term  $(Y_R)_{ij} \mathbf{16}_i \mathbf{16}_j \overline{\mathbf{126}}_H$ , the same term will generate the matrix  $M_T$ , since  $\overline{\mathbf{126}}_H$  contains the triplet  $T$ . In the very simplest  $SO(10)$  models with one  $\overline{\mathbf{126}}_H$ , therefore, one would expect  $M_\nu^{II} = M_T = c M_R (v/M_G)^2$ , where  $c \sim 1$  [4]. The reason that the vacuum expectation value (VEV) of  $T$  is of order  $v^2/M_G$  is simple. If  $T$  has a conjugate field  $\overline{T}$ , then one expects that in the superpotential there will be terms of the form  $M_T \overline{T} T + \overline{T} H_u H_u$ , where  $M_T \sim M_G$ . Integrating out  $\overline{T}$  gives  $-F_T^* = M_T \langle T \rangle + \langle H_u \rangle^2 = 0$ .

The type I and type II see-saw formulas can be understood as arising from block-diagonalizing the complete mass matrix of the neutrinos and anti-neutrinos:

$$\mathcal{L}_{\nu \text{ mass}} = (\nu_i, N_i^c) \begin{pmatrix} (M_T)_{ij} & (M_N)_{ij} \\ (M_N^T)_{ij} & (M_R)_{ij} \end{pmatrix} \begin{pmatrix} \nu_j \\ N_j^c \end{pmatrix}, \quad (2)$$

with  $M_R \sim M_G$ ,  $M_N \sim v$ ,  $M_T \sim v^2/M_G$ , giving  $M_\nu = M_\nu^I + M_\nu^{II} = -M_N M_R^{-1} M_N^T + M_T$ , neglecting terms higher order in  $v/M_G$ .

In the alternative where  $B - L$  breaking is produced by the expectation values of  $\overline{\mathbf{16}}$  and  $\mathbf{16}$  of Higgs fields,  $M_R$  arises from an effective operator of the form

$$\mathcal{O}_R = (Y_R)_{ij} \mathbf{16}_i \mathbf{16}_j \overline{\mathbf{16}}_H \overline{\mathbf{16}}_H / M_G. \quad (3)$$

Such an operator comes from the diagram in Fig. 2. The VEV  $\langle \mathbf{1}(\overline{\mathbf{16}}_H) \rangle \equiv \Omega$  and the mass matrix  $M_{mn}$  in that diagram are  $O(M_G)$ .  $\mathbf{M}(\mathbf{N})$  denotes an

$SU(5)$   $\mathbf{M}$  representation contained in an  $SO(10)$   $\mathbf{N}$ . We assume for simplicity that the fields  $S_m$  in Fig. 2 are singlets under  $SO(10)$  as well as  $SU(5)$ , though it would change nothing in the later discussion if they were in other representations of  $SO(10)$ .

In addition to the couplings  $M_{mn}\mathbf{1}_m\mathbf{1}_n$  and  $F_{im}\mathbf{16}_i\mathbf{1}_m\overline{\mathbf{16}}_H$  involved in Fig. 2, there is the Dirac mass term  $(M_N)_{ij}\nu_i N_j^c$  that comes from Yukawa terms of the form  $\mathbf{16}_i\mathbf{16}_j H$ , where  $H \subset \overline{\mathbf{16}} \times \overline{\mathbf{16}}$ . The full mass matrix of the neutrinos and anti-neutrinos (which now include  $\nu_i$ ,  $N_i^c$ , and  $S_m$ ) is

$$\mathcal{L}_{\nu \text{ mass}} = (\nu_i, N_i^c, S_m) \begin{pmatrix} 0 & (M_N)_{ij} & 0 \\ (M_N^T)_{ij} & 0 & F_{in}\Omega \\ 0 & F_{mj}^T\Omega & M_{mn} \end{pmatrix} \begin{pmatrix} \nu_j \\ N_j^c \\ S_n \end{pmatrix}, \quad (4)$$

where  $i, j = 1, 2, 3$  and  $m, n = 1, \dots, N$ , and  $N$  is the number of species of singlets  $S_m$ . It is easy to show that the effective mass matrix  $M_\nu$  of the light neutrinos is given, up to negligible corrections higher order in  $v/M_G$ , by

$$M_\nu = -M_N(F\Omega M^{-1} F^T\Omega)^{-1} M_N^T. \quad (5)$$

In other words, one has the usual type I see-saw formula with

$$M_R = (F\Omega)M^{-1}(F^T\Omega). \quad (6)$$

In such a model there is no type II see-saw contribution, as the  $\overline{\mathbf{16}}$  and  $\mathbf{16}$  do not contain a weak-triplet Higgs field. However, another type of new see-saw contribution can arise as we will now see.. So far, we have only taken into account the VEV of the  $\mathbf{1}(\overline{\mathbf{16}})$  component of the  $\overline{\mathbf{16}}_H$ , which we called  $\Omega$ . However, there is a weak doublet in the  $\mathbf{5}(\overline{\mathbf{16}})$  that can have a weak-scale VEV, which we shall call  $u$ . There is no a priori reason why  $u$  should vanish. If it does not, then the term  $F_{im}\mathbf{16}_i\mathbf{1}_m\overline{\mathbf{16}}_H$  not only produces the  $O(M_G)$  mass term  $F_{im}(N_i^c S_m)\Omega$ , but an  $O(v)$  mass term  $F_{im}(\nu_i S_m)u$ . Eq. (4) then becomes

$$\mathcal{L}_{\nu \text{ mass}} = (\nu_i, N_i^c, S_m) \begin{pmatrix} 0 & (M_N)_{ij} & F_{in}u \\ (M_N^T)_{ij} & 0 & F_{in}\Omega \\ F_{mj}^T u & F_{mj}^T\Omega & M_{mn} \end{pmatrix} \begin{pmatrix} \nu_j \\ N_j^c \\ S_n \end{pmatrix}, \quad (7)$$

This can be simplified by a rotation in the  $\nu_i N_i^c$  plane by angle  $\tan^{-1}(u/\Omega)$ :  $\nu'_i = (\nu_i - \frac{u}{\Omega} N_i^c)/\sqrt{1 + (u/\Omega)^2}$ ,  $N_i^{c'} = (N_i^c + \frac{u}{\Omega} \nu_i)/\sqrt{1 + (u/\Omega)^2}$ , which has

the effect of eliminating the  $\nu S$  entries. It also replaces the 0 in the  $\nu\nu$  entry by

$$M_\nu^{III} = -(M_N + M_N^T) \frac{u}{\Omega}, \quad (8)$$

neglecting, as always, terms higher order in  $v/M_G$ . Otherwise, the resulting matrix has the same form as Eq. (4). Therefore, the full result for  $M_\nu$  is given by the sum of Eqs. (5) and (8).

The relative size of the two contributions to  $M_\nu$  is model dependent. Since  $M_N$  is related to the up quark mass matrix  $M_U$  by  $SO(10)$ , one would expect the entries for the first and second families to be very small compared to  $v$ . Consequently, due to the fact that  $M_N$  comes in squared in  $M_\nu^I$  but only linearly in  $M_\nu^{III}$ , the latter should dominate, except perhaps for the third family.  $M_\nu^{III}$  would also dominate if the elements of  $M_{mn}$  were small compared to  $\Omega \simeq M_G$ , as Eqs. (5) and (6) show.

That  $M_\nu^{III}$  dominates is an interesting possibility, as remarkably predictive  $SO(10)$  models of quark and lepton masses would then be constructable. Usually the most one can achieve in models where  $M_\nu$  is given by the type I see-saw formula is predictions for the mass matrices of the up quarks, down quarks, and charged leptons ( $M_U$ ,  $M_D$ ,  $M_L$ ), and for the Dirac mass matrix of the neutrinos ( $M_N$ ), since these four matrices are intimately related to each other by symmetry. (For example in the “minimal  $SO(10)$  model” they all come from one term  $Y_{ij} \mathbf{16}_i \mathbf{16}_j \mathbf{10}_H$  and have exactly the same form.) However, sharp predictions for neutrino masses and mixings are hard to achieve because of the difficulty in constraining the form of  $M_R$ , which comes from different terms. On the other hand, if the type III see-saw contributions are dominant, then the matrix  $M_R$  is irrelevant; a knowledge of  $M_N$  and  $M_L$  is sufficient to determine the neutrino mass ratios and mixing angles.

In this letter we will not be so ambitious. Rather we will look at a version of the type III see-saw that is less predictive but still has certain attractive features. In the foregoing, we assumed that there was only a single  $\overline{\mathbf{16}}$  of Higgs fields that contributed to neutrino masses. If there is more than one, then their coupling to neutrinos comes from the term  $\sum_{aim} F_{im}^a (\mathbf{16}_i \mathbf{1}_m) \overline{\mathbf{16}}_{Ha}$ , which contains  $\sum_{im} F_{im} (N_i^c S_m) \Omega + \sum_{im} F'_{im} (\nu_i S_m) u$ , where  $F_{im} \equiv \sum_a F_{im}^a \Omega_a / \Omega$ ,  $F'_{im} \equiv \sum_a F_{im}^a u_a / u$ ,  $\Omega \equiv (\sum_a \Omega_a^2)^{1/2}$ , and  $u \equiv (\sum_a u_a^2)^{1/2}$ . Then Eq. (7) is modified by having Yukawa matrices in the  $\nu S$  and  $N^c S$  blocks that are no longer proportional to each other. It is then not possible to null out the  $\nu S$  block of  $M_\nu$  by a simple flavor-independent rotation by angle  $\tan^{-1}(u/\Omega)$ , as

in the special case discussed above. Consequently, the effective light-neutrino mass matrix is more complicated. In the most general case it can be written,  $M_\nu = -\tilde{M}_N M_R^{-1} \tilde{M}_N^T + (F'u)M^{-1}(F'^T u)$ , where  $M_R$  is given by Eq. (6) as before, and  $\tilde{M}_N \equiv M_N + (F'u)M^{-1}(F'^T \Omega)$ . However, a great simplification results if one assumes that the number of species of singlet fermions  $S_m$  is three, i.e. one for each family. (If there were less than three, not all the  $N_i^c$  would get superlarge mass, and some of the light neutrinos would have masses of order  $v$ .) With three species of  $S_m$ , the matrices  $F$  and  $F'$  are (generally) invertible and then  $M_\nu = M_\nu^I + M_\nu^{III}$ , where  $M_\nu^I$  is given as before by Eq. (5) and

$$M_\nu^{III} = -(M_N H + H^T M_N^T) \frac{u}{\Omega}, \quad H \equiv (F' F^{-1})^T. \quad (9)$$

In this the generalized type III see-saw formula, the dimensionless  $3 \times 3$  matrix  $H$  introduces many unknown parameters, more indeed than does  $M_R$  in the type I see-saw. However, as we shall now show by an example, in  $SO(10)$  models it may be easier to obtain a realistic pattern of neutrino masses and mixings without fine-tuning of parameters in the generalized type III see-saw than in the type I see-saw.

The  $SO(10)$  model of Ref. [5] gives an excellent fit to the quark masses and mixings and the charged lepton masses, fitting 13 real quantities with 8 real parameters. This fit uniquely determines the neutrino Dirac mass matrix (at the unification scale) to be

$$M_N = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix} m_U, \quad (10)$$

where  $m_U \cong m_t$ ,  $\eta \cong m_u^0/m_t^0 \cong 0.6 \times 10^{-5}$ , and  $\epsilon \cong 3\sqrt{m_e^0/m_t^0} \cong 0.14$ . (Superscripts here refer to quantities evaluated at  $M_G$ .) In this model there is a very large (namely  $\tan^{-1} 1.8$ ) contribution to the atmospheric neutrino mixing angle coming from the charged lepton mass matrix  $M_L$ , which is completely known. However, as  $M_R$  is not known, it is impossible to predict the neutrino mass ratios and the other neutrino mixing angles (or even the atmospheric angle precisely) within the framework of the type I see-saw. Nevertheless, one can ask whether what we know about these neutrino masses and mixings can be accommodated in the model with a reasonable form for

$M_R$ . Parametrizing that matrix by  $(M_R^{-1})_{ij} = a_{ij}m_R^{-1} = a_{ji}m_R^{-1}$ , the type I see-saw formula gives

$$M_\nu = \begin{pmatrix} a_{11}\eta^2 & -a_{13}\epsilon\eta & (a_{13} + a_{12}\epsilon)\eta \\ -a_{13}\epsilon\eta & a_{33}\epsilon^2 & -(a_{33} + a_{23}\epsilon)\epsilon \\ (a_{13} + a_{12}\epsilon)\eta & -(a_{33} + a_{23}\epsilon)\epsilon & a_{33} + 2a_{23}\epsilon + a_{22}\epsilon^2 \end{pmatrix} m_U^2/m_R. \quad (11)$$

Neglecting the relatively small first row and column, the condition that the ratio  $m_2/m_3$  of the two heaviest neutrino masses be equal to some value  $r$  is that

$$a_{22}a_{33} - a_{23}^2 \cong \frac{r}{(1+r^2)^2} (a_{33}\frac{1}{\epsilon^2} + 2a_{23}\frac{1}{\epsilon} + a_{22} + a_{33})^2. \quad (12)$$

It is evident that  $r$  naturally is of order  $\epsilon^4 \approx 4 \times 10^{-4}$ . For  $r$  to be of order  $\epsilon^0$  (as indicated by experiment, which gives  $r \approx 1/6$ ) the elements must be somewhat “tuned”. For example, setting  $a_{23}/a_{33} = p\epsilon^{-1} + O(\epsilon^0)$  and  $a_{22}/a_{33} = q\epsilon^{-2} + O(\epsilon^{-1})$ , Eq. (12) gives the condition  $1 + 2p + q = 0$ . In other words, not only must the 23 block of  $M_R$  have a hierarchy that is correlated with the hierarchy of the 23 block of  $M_N$ , but it must also satisfy a non-trivial numerical relation among its elements. This kind of mild fine-tuning of the 23 block of  $M_R$  is typically required in  $SO(10)$  models with the type I see-saw mechanism [6]

It can be seen from Eq. (11) that to fit the LMA solar solution  $a_{11} \leq \epsilon^2/\eta^2$ ,  $a_{12} \leq \epsilon/\eta$ , and  $a_{13} \sim \epsilon^2/\eta$ . Thus the correlation between the hierarchies of  $M_R$  and  $M_N$  extends also to the first family.

By contrast, a satisfactory pattern of neutrino masses and mixings can be achieved without any fine-tuning in this model if the type III see-saw mechanism dominates. There are two interesting cases. Suppose, first, that all the elements of  $F$  are of the same order, and likewise for  $F'$ . Then all the elements of  $H \equiv (F'F^{-1})^T$  will be of order one. From Eq. (9), neglecting terms of order  $\eta$ ,

$$M_\nu = \begin{pmatrix} 0 & \epsilon H_{31} & H_{31} - \epsilon H_{21} \\ \epsilon H_{31} & 2\epsilon H_{32} & H_{32} + \epsilon(H_{33} - H_{22}) \\ H_{31} - \epsilon H_{21} & H_{32} + \epsilon(H_{33} - H_{22}) & 2(H_{33} - \epsilon H_{23}) \end{pmatrix} \frac{m_U u}{\Omega}. \quad (13)$$

Here it is clear that without any fine-tuning  $|r|$  ( $\equiv |m_2/m_3|$ ) is somewhat less than one, as desired. More precisely:  $-r/(1+r^2) \cong \frac{1}{4}(H_{32}/H_{33})^2 + O(\epsilon)$ .

Moreover, the LMA solution naturally emerges. For  $U_{e3}$  to be consistent with present limits,  $\epsilon H_{21}$  must approximately cancel  $H_{31}$  in the 13 and 31 elements of  $M_\nu$ . However, all the other elements of  $H$  can be of order one.

Note that a satisfactory pattern of light neutrino masses and mixings emerges with *no hierarchy* among the superheavy neutrinos, which all have masses of order  $M_G$ , something that is impossible in the type I see-saw. This is an attractive possibility, but would create problems for leptogenesis [7].

A second interesting case is that  $F$  and  $F'$  both have the form

$$F, F' \sim \begin{pmatrix} \eta/\epsilon & \eta/\epsilon & \eta/\epsilon \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (14)$$

as might arise naturally if the first family had a different abelian family charge than the others. Then, by Eq. (9),  $M_\nu$  has the form

$$M_\nu \sim \begin{pmatrix} \eta & \epsilon & \epsilon \\ \epsilon & \epsilon & 1 \\ \epsilon & 1 & 1 \end{pmatrix} \frac{m_U u}{\Omega}, \quad (15)$$

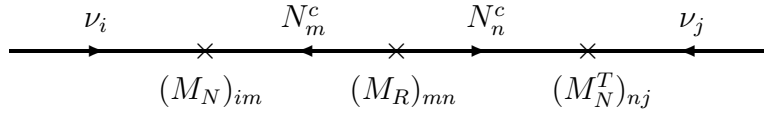
that is, the same form as the previous case, except that  $U_{e3}$  is automatically of order  $\epsilon$ . In this case, the superheavy neutrinos consist of one pseudo-Dirac neutrino with mass  $O((\eta/\epsilon)M_G) \sim 10^{12}$  GeV and two pseudo-Dirac neutrinos with mass of order  $M_G$ .

## Figure Captions

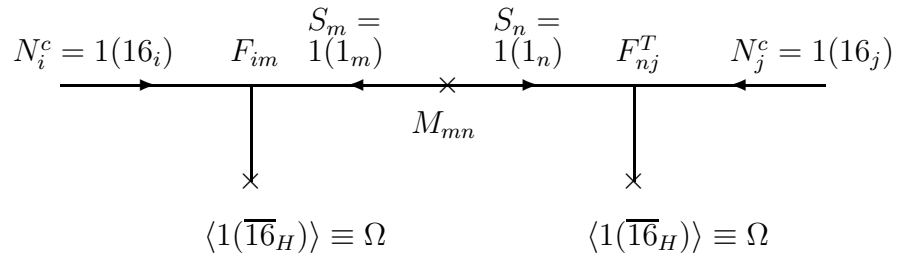
Fig. 1: Diagram that gives the light neutrinos type I see-saw masses of order  $v^2/M_G$ .

Fig. 2: Diagram that produces the effective operator  $\mathbf{16}_i \mathbf{16}_j \overline{\mathbf{16}}_H \overline{\mathbf{16}}_H / M_G$ , which generates  $M_R$ .





**Fig. 1**



**Fig. 2**

## References

- [1] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, Proceedings of the Workshop. Stony Brook, New York, 1979, ed. P. van Nieuwenhuizen and D.Z. Freedman (North-Holland, Amsterdam, 1979), p.315; T. Yanagida, in *Proc. Workshop on Unified Theory and the Baryon Number of the Universe*, Tsukuba, Japan, 1979, ed. O. Sawada and A. Sugamoto (KEK Report No. 79-18, Tsukuba, 1979), p.95; R.N. Mohapatra and G. Senjanovic, *Phys. Rev. Lett.***44**, 912 (1980); S.L. Glashow, in *Quarks and Leptons*, Cargese (July 9-29, 1979), ed. M. Levy et al. (Plenum, New York, 1980), p.707.
- [2] M. Shiozawa (Super-Kamiokande Collaboration), talk given at the Int. Conf. on Neutrino Physics and Astrophysics, “Neutrino ’02”, May 25-30, 2002, Munich, Germany.
- [3] Super-Kamiokande Collaboration, nucl-ex/0309004.
- [4] G. Lazarides, Q. Shafi, and C. Wetterich, *Nucl. Phys.* **B181**, 287 (1981), R.N. Mohapatra and G. Senjanovic, *Phys. Rev.* **D23**, 165 (1981).
- [5] C.H. Albright and S.M. Barr, *Phys. Lett.* **B452**, 287 (1999); C.H. Albright, K.S. Babu, and S.M. Barr, *Phys. Rev. Lett.***81**, 1167 (1998).
- [6] S.M. Barr and I. Dorsner, *Nucl. Phys. B* **585**, 79 (2000).
- [7] W. Buchmüller, P. Di Bari, and M. Plümacher, hep-ph/0302092; E. Akhmedov, M. Frigerio, and A.Yu. Smirnov, hep-ph/0305322.